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# Proof of Straley's Argument for Bootstrap Percolation

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We prove that $P_c = 1$ for bootstrap percolation with large void instabilities (in particular, if $m = 3$ on the square lattice).

**KEY WORDS:** Bootstrap percolation; large voids; contours.

Bootstrap percolation is a modified site percolation process where each site, after being occupied with probability $P$, remains occupied only if at least $m$ of its neighboring sites are occupied. If the number of neighbors is less, the site becomes empty (or "culled") and this process continues until each occupied site has at least $m$ neighbors.

This model was introduced in Ref. 1. For more information about bootstrap percolation see Refs. 2–6. We note that in Ref. 6 also a class of models that is dual to bootstrap percolation was introduced, so-called diffusion percolation. The following argument also applies to this case.

If $m = 3$ on the square lattice (or $m = 4$ on the triangular lattice or $m = 5$ on the simple cubic lattice), the allowed finite sets of empty sites have fixed, rigid shapes (rectangles, truncated equilateral triangles, and rectangular prisms, respectively). In this case Straley (unpublished; see Kogut and Leath[(2)]) argued that $P_c = 1$ because of "large void instabilities." Here we give a simple proof of this assertion . We write down the proof for the square lattice with $m = 3$; the other cases are analogous.

The strategy of our proof is to show that with positive probability (and hence, because of ergodicity, with probability one) there exists a large empty square that is not surrounded by any rectangular contour of occupied sites. If one successively empties ("culls") all the sites that have

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less than $m$ neighbors, such a square will end up by eating the whole lattice.

*Proof.* Take a fixed square $\square$ of $N \times N$ sites and let $1 > \varepsilon > 0$. Subadditivity of probability gives

$$\text{Prob}(\square \text{ is surrounded by an occupied rectangle})$$
$$< \sum_{n=4N}^{\infty} P^n \,\#(\text{rectangles around } \square \text{ of length } n)$$
$$< \sum_{n=4N}^{\infty} P^n \,\text{Polyn}(n) < \varepsilon \tag{1}$$

if $N$ (dependent on $P$ and $\varepsilon$) is large enough.

*Remark.* The number of rectangles around the origin is polynomially bounded, because (a) the number of shapes for a rectangle of length $2n$ equals $n-1$, and (b) the number of rectangles of length $2n$ and fixed shape surrounding the origin is smaller than or equal to $(n-1)^2$.

Hence

$$\text{Prob}(\square \text{ is not surrounded by any occupied rectangle})$$
$$> 1 - \varepsilon \tag{2}$$
$$\text{Prob}(\square \text{ is empty and not surrounded by any occupied rectangle})$$
$$> (1-P)^{N^2}(1-\varepsilon) \gneq 0 \tag{3}$$

The occurrence of such a $\square$ somewhere is a translation-invariant event and by the ergodic theorem[7] it has probability one. (In fact, using only the rigidity, the occupied rectangle need not be occupied at the corner points, but this does not change the argument.[2]) Hence, for any $P < 1$ the lattice becomes empty for this percolation process and thus $P_c = 1$.

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[2] This remark might apply to related models; see, e.g., Ref. 8.

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